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ON AN EFFECTIVE ALGORITHM FOR MINIMIZING GENERALIZED TREFFTZ FUNCTIONALS OF LINEAR ELASTICITY THEORY*

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The problem of minimizing the generalized Trefftz functionals of three-dimensional elasticity theory results in a minimax problem for the Lagrangian. An algorithm is proposed for searching for the saddle point in coordinate functions not subjected to any constraints in the domain and on the boundary (this is the efficiency of the algorithm). The convergence of the approximate solution is investigated.

The Trefftz variational method /1/ is convenient for solving boundary value problems of mathematical physics in that the dimensionality of the problem being solved is reduced because of its reduction to the solution of equations defined on the domain boundary. At the same time, when constructing the solution using the Ritz process, say, the coordinate functions should be selected so that they satisfy the differential equation of the boundary value problem in the domain, which is a serious constraint. An approach is proposed below that uses Lagrange multipliers to reduce this constraint when minimizing the generalized Trefftz functionals of the fundamental boundary value problems of linear elasticity theory. The results obtained can also be used to minimize the classical Trefftz functionals of the boundary value problems of mathematical physics /1/.

Generalized Trefftz functionals were constructed in /2, 3/ for the fundamental problems of linear elasticity theory with continuous and discontinuous elasticity coefficients. The functionals are minimized in solutions (ordinary or generalized) for the linear equilibrium equation for an elastic medium in displacements. Assuming the existence of a coordinate system of functions satisfying the equilibrium equation (in the generalized sense) in /4/, the Ritz process was investigated for solving problems to minimize the generalized Trefftz functionals in an example of the second boundary value problem of three-dimensional elasticity theory. The practical construction of the above-mentioned coordinate system is a fairly complex problem. At the same time, the differential equation of the boundary value problem in whose solutions the minimum of the functionals is sought, can be considered as a linear constraint in the problem of minimizing the Trefftz functionals. Then such a minimization problem with linear constraints can be reduced to the minimax problem of a certain Lagrangian (by using reciprocity theory).

1. The notation in /2,3/ is used henceforth. Let $\Phi(u)$ be a generalized Trefftz functional of one of the fundamental boundary value problems of linear elasticity theory with the domain of definition

$$D_1(\Phi) = \{u \in W_2^2(G) \mid Au \in L_2(G),\ Au = K\}$$

which can be extended as follows:

$$D_2(\Phi) = \left\{u \in W_2^1(G) \,\Big|\, 2\int_G W^\cdot(u, v)\, dG - \int_S t(u)\, v\, ds = \int_G Kv\, dG,\quad \forall v \in W_2^1(G)\right\}.$$

Here $u \in D_2(\Phi)$ is the generalized solution of the equilibrium equation $Au = K$ in the

*Prikl. Matem. Mekhan., 49, 2, 292-298, 1985 domain of the elastic medium $G \subset E_3$ with boundary S.

It is evident that if $u \in D_1(\Phi)$. then also $u \in D_2(\Phi)$. i.e., $D_1(\Phi) \subset D_2(\Phi)$. Here and henceforth, u, v are vector functions, K is a given vector of the mass forces, $W_2^1(G)$, $W_2^2(G)$ are the standard notation for the Sobolev classes of functions, $L_2(G)$ is the Hilbert space of functions, square-summable in G.

It is well-known /1, 3/ that the minimum of the Trefftz functionals is reached in the energy solution u_0 of the boundary value problem, and this minimum equals $\Phi(u_0) = |u_0|^2$, where $|\cdot|$ is the energy norm (i.e., u_0 is an element realizing the minimum of the energy functional of the boundary value problem /1/).

The linear constraint $Au = K$ of the problem of finding $\inf \Phi(u)$ for $u \in D_1(\Phi)$ can be reduced by using the Lagrange multiplier method /5, 6/. We define the set of such vectors $\lambda \in L_2(G)$ such that

$$\sup_{\lambda \in L_2(G)} \int_G \lambda (Au - K)\, dG = \begin{cases} 0, & u \in D_1(\Phi) \\ +\infty, & u \notin D_1(\Phi). \end{cases}$$

Then the problem of determining $\inf \Phi(u)$ for $u \in D_1(\Phi)$ reduces to an equivalent problem (see Sec.2) of determining

$$\inf_{u \in W_2^2(G)} \sup_{\lambda \in L_2(G)} \left[\Phi(u) - 2\int_G \lambda (Au - K)\, dG\right] \tag{1.1}$$

which is later called direct. Therefore, the function (Lagrangian)

$$L(u, \lambda) = \Phi(u) - 2\int_G \lambda (Au - K)\, dG: W_2^2(G) \times L_2(G) \to R$$

has been defined.

The problem of finding

$$\sup_{\lambda \in L_2(G)} \inf_{u \in W_2^2(G)} L(u, \lambda) \tag{1.2}$$

is dual to problem (1.1). Below (Sec.2) the existence is proved for the saddle point $\{u_0, \lambda_0\} \in W_2^2(G) \times L_2(G)$ of the Lagrangian $L(u, \lambda)$, one of whose arguments is u_0.

From the variational equations that express the necessity and sufficiency of the conditions that two partial derivatives of the function $L(u, \lambda)$ vanish at the saddle point $\{u_0, \lambda_0\}$, we obtain

$$2\Phi(u_0, v) - 2\int_G \lambda_0 Av\, dG = 0, \quad \forall v \in W_2^2(G) \tag{1.3}$$

$$\int_G \lambda (Au_0 - K)\, dG = 0, \quad \forall \lambda \in L_2(G). \tag{1.4}$$

An interpretation of the Lagrange multiplier λ_0 can be obtained from (1.3). To do this, we use the expressions of the bilinear functionals $\Phi_i(u, v)$ of the corresponding fundamental boundary value problems /3/: the first ($i = 1$), second ($i = 2$), and third ($i = 3$)

$$\Phi_1(u, v) = I(u, v) + 2(u, v)_{1/2, S}$$

$$\Phi_2(u, v) = I(u, v) + \frac{1}{\alpha}(U, t(u))_{0, S}(U, t(v))_{0, S} - (u, t(v))_{0, S} - (v, t(u))_{0, S}$$

$$\Phi_3(u, v) = I(u, v) + \frac{1}{\alpha}(U, t(u))_{0, S_2}(U, t(v))_{0, S_2} - (u, t(v))_{0, S_1} - (v, t(u))_{0, S_2} - \alpha(u, v)_{1/2, S_1}.$$

Here /3/

$$I(u, v) = 2\int_G W(u, v)\, dG$$

$W(u)$ is a positive-definite quadratic form in linear elasticity theory /1/, $(\,,)_{0,S}$, and $(\,,)_{1/2,S}$ are scalar products in the Hilbert spaces $L_2(S)$, $W_2^{1/2}(S)$ ($W_2^{1/2}(S)$ is the Sobolev-Slobodetskii space of traces on S), U is a certain fixed displacement vector, $t(u)$ is a surface stress vector associated with the displacement vector u, and α is a certain positive constant. When the boundary conditions of the fundamental problems are satisfied $u_0|_S = 0$ for the first; $t(u_0)|_S = 0$ for the second, $S = S_1 \cup S_2$, $u_0|_{S_1} = 0$, $t(u_0)|_{S_2} = 0$ for the third, by using the Betti formula /1/

$$2\int_G W(u,v)\,dG - \int_S ut(v)\,ds = \int_G uAvdG$$

we obtain the following equation for all the fundamental problems

$$\Phi_i(u_0, v) = \int_G u_0 AvdG \quad (i=1,2,3), \quad \forall v \in W_2^2(G).$$

Then it follows from (1.3) that

$$\int_G u_0 AvdG = \int_G \lambda_0 AvdG, \quad \forall v \in W_2^2(G).$$

Hence it follows that the Lagrange multiplier λ_0 has the meaning of the elastic displacements vector u_0.

2. The saddle point $\{u_0, \lambda_0\}$ of the Lagrangian $L(u, \lambda)$ is determined by the condition /5/

$$L(u_0, \lambda) \leq L(u_0, \lambda_0) \leq L(u, \lambda_0), \ \forall u \in W_2^2(G), \ \lambda \in L_2(G).$$

The function $L(u, \lambda)$ defined on $W_2^2(G) \times L_2(G)$ and taking finite values has a saddle point $\{u_0, \lambda_0\}$ on $W_2^2(G) \times L_2(G)$ if and only if (/5/, p. 172)

$$L(u_0, \lambda_0) = \inf_{u \in W_2^2(G)} \sup_{\lambda \in L_2(G)} L(u,\lambda) = \sup_{\lambda \in L_2(G)} \inf_{u \in W_2^2(G)} L(u,\lambda). \tag{2.1}$$

Let us prove this relationship. From $L(u, \lambda) = \Phi(u) - 2\int_G \lambda(Au - K)\,dG$ for $u = u_0$ and $\forall\lambda$ it follows that $L(u_0, \lambda_0) = \Phi(u_0) = |u_0|^2$.

We establish by direct substitution that

$$\inf_{u \in W_2^2(G)} \sup_{\lambda \in L_2(G)} L(u,\lambda) = |u_0|^2.$$

Indeed

$$\sup_{\lambda \in L_2(G)} \left[\Phi(u) - 2\int_G \lambda(Au - K)\,dG\right] = \begin{cases} +\infty, & u \overline{\in} D_1(\Phi) \\ \Phi(u), & u \in D_1(\Phi) \end{cases}.$$

Therefore, we obtain what is required

$$\inf_{u \in W_2^2(G)} \sup_{\lambda \in L_2(G)} L(u,\lambda) = \inf_{u \in W_2^2(G)} \begin{Bmatrix} +\infty, & u \overline{\in} D_1(\Phi) \\ \Phi(u), & u \in D_1(\Phi) \end{Bmatrix} = \inf_{u \in D_1(\Phi)} \Phi(u) = \Phi(u_0) = |u_0|^2.$$

We will also prove that

$$\sup_{\lambda \in L_2(G)} \inf_{u \in W_2^2(G)} L(u,\lambda) = |u_0|^2.$$

For a certain fixed $\lambda \in L_2(G)$ the solution u_λ of the problem of determining $\inf L(u, \lambda)$ for $u \in W_2^2(G)$ is a solution of the equation $\mathrm{grad}_u L(u_\lambda, \lambda) = 0$ (see (1.3)), i.e.

$$2\Phi(u_\lambda, v) - 2\int_G \lambda AvdG = 0, \quad \forall v \in W_2^2(G). \tag{2.2}$$

It therefore follows that for $v = u_\lambda$

$$\Phi(u_\lambda) = \int_G \lambda Au_\lambda dG, \quad \forall\lambda \in L_2(G).$$

We evaluate the lower bound of $L(u, \lambda)$ (for fixed λ)

$$L(u_\lambda, \lambda) = \Phi(u_\lambda) - 2\int_G \lambda(Au_\lambda - K)\,dG = \int_G \lambda Au_\lambda dG - 2\int_G \lambda(Au_\lambda - K)\,dG = -\int_G \lambda Au_\lambda dG + 2\int_G \lambda KdG.$$

Then the dual problem of (1.2) reduces to the minimization problem

$$\sup_{\lambda \in L_2(G)} L(u_\lambda, \lambda) = \sup_{\lambda \in L_2(G)} \left(-\int_G \lambda Au_\lambda dG + 2\int_G \lambda KdG\right) = -\inf_{\lambda \in L_2(G)} \left(\int_G \lambda Au_\lambda dG - 2\int_G \lambda KdG\right) \tag{2.3}$$

where u_λ is determined from (2.2).

If $u_\lambda = u_0$ and $\lambda = \lambda_0 = u_0$ (see Sec.1), then the expression

$$\int_G u_0 A u_0 dG - 2\int_G u_0 K dG = -\int_G u_0 A u_0 dG = -|u_0|^2$$

is an energy functional (/1/, p. 90) defined on the elastic displacments vector. We therefore also obtain from (2.3) that

$$\sup_{\lambda} \inf_{u} L(u, \lambda) = |u_0|^2$$

(see also /6/,pp.37, 42). Therefore, the relationship (2.1) is proved.

It follows from (1.4) that the argument u_0 of the saddle point (the element minimizing the generalized Trefftz functional $\Phi(u)$) satisfies the constraint of the problem $Au_0 = K$. It can be confirmed that the function u_λ, the solution of (2.2) for each fixed λ, also satisfies the constraint $Au_\lambda = K$. Indeed (see (1.4))

$$\operatorname{grad}_\lambda L(u_\lambda, \lambda) = -2\int_G Au_\lambda \mu dG + 2\int_G K\mu dG = 0, \quad \forall \mu \in L_2 . \tag{2.4}$$

It hence follows that $Au_\lambda = K$.

Remark. Since the lower (upper) bound is achieved by virtue of what was proved in (2.1), then inf (sup) can be replaced in (2.1) by min (max).

Therefore the problem of finding the minimum of generalized Trefftz functionals in solutions of the equilibrium equation of an elastic medium reduces to solving an equivalent problem resulting from the dual formulation of the problem on the maximin of the Lagrangian. The equivalent problem reduces to solving the variational equations (2.2) and (2.4). The efficiency of the approach elucidated is, from the viewpoint of solving boundary value problems, that in constructing the solutions of (2.2) and (2.4) constraints are not imposed on the basis functions in the sense of satisfying boundary conditions (which are satisfied automatically upon minimizing the Treffts functionals /1/) and the equilibrium equation in the domain.

3. We elucidate as possible algorithm to search for the saddle point of the Lagrangian $L(u, \lambda)$. The algorithm is based on using (2.2) and (2.4).

Let $\{\varphi_i\}_{i=1}^{i=\infty}$ be a system of fairly smooth functions (for the validity of the constructions presented above it is evidently sufficient that the functions φ_i belong to the class $W_2^2(G)$). Later, completeness of the system $\{\varphi_i\}$ is required only in $L_2(G)$ (i.e., in the sense of convergence in the mean) for the convergence of the approximate solution. In addition to the above, no other constraints are imposed on the function φ_i in the domain G or on the boundary S.

We form two sequences of linear combinations of linearly independent functions φ_i

$$u_k = \sum_{i=1}^{k} a_i\varphi_i, \quad \lambda_n = \sum_{j=1}^{n} b_j\varphi_j \tag{3.1}$$

(in particular, there can be $k = n$), where a_i, b_j are constants to be determined.

Obviously, Eq.(2.2) is also satisfied for all functions $v_k \in W_2^2(G)$ of the form

$$v_k = \sum_{m=1}^{k} \alpha_m\varphi_m$$

where α_m are arbitrary. Then the following relationship holds:

$$\Phi(u_\lambda, \varphi_m) - \int_G \lambda A\varphi_m dG = 0, \quad \forall\varphi_m, \quad m = 1, 2, \ldots, k . \tag{3.2}$$

For each fixed $\lambda = \lambda_n$ an approximate solution of the form u_k for (3.2) is written in the form

$$u_{\lambda_n} \equiv u_k(\lambda_n) = \sum_{i=1}^{k} a_{in}\varphi_i = \sum_{i=1}^{k} a_i \left(\sum_{j=1}^{n} b_j c_{jm}\right)\varphi_i$$

where the dependence (c_{jm} are certain numbers, see below)

$$a_i \to \sum_{j=1}^{n} b_j c_{jm} \quad (i = 1, 2, \ldots, k)$$

is determined from the system of linear equations

$$\sum_{i=1}^{k} a_i\Phi(\varphi_i, \varphi_m) - \sum_{j=1}^{n} b_j \int_G \varphi_j A\varphi_m dG = 0, \quad m = 1, 2, \ldots, k \tag{3.3}$$

Similarly, (2.4) is also satisfied for all functions $\mu_n \in L_2(G)$ of the form

$$\mu_n = \sum_{l=1}^{n} \beta_l \varphi_l$$

where β_l are arbitrary. Then we have

$$\int_G Au_\lambda \varphi_l dG - \int_G K\varphi_l dG = 0, \quad \forall \varphi_l, \quad l = 1, 2, \ldots, n \tag{3.4}$$

For the approximations

$$u_{\lambda_n} = \sum_{i=1}^{k} a_i \left(\sum_{j=1}^{n} b_j c_{jm}\right) \varphi_i$$

we obtain from (3.4) a system of linear equations to determine b_j

$$\sum_{i=1}^{k} a_i \left(\sum_{j=1}^{n} b_j c_{jm}\right) \int_G A\varphi_i \varphi_l dG - \int_G K\varphi_l dG = 0, \quad l = 1, 2, \ldots, n \tag{3.5}$$

Thus, (3.3) and (3.5) jointly comprise a system of linear equations to determine the constants a_i and b_j in the expansion (3.1) determining the approximate value of the saddle point of the Lagrangian $L(u, \lambda)$.

Here the matrix $\Phi(\varphi_i, \varphi_m)$ of the system (3.3) to determine the dependence

$$a_i \to \sum_{j=1}^{n} b_j c_{jm} \quad (i = 1, 2, \ldots, k)$$

is symmetric and positive-definite by virtue of the estimate (see /3/)

$$\Phi(u_\lambda) \geqslant c \| u_\lambda \|^2_{W_2^1(G)}, \quad c > 0 \tag{3.6}$$

(the dependence mentioned will evidently be linear). The system of Eqs.(3.5) is also solvable uniquely because of the positive-definiteness of the operator A, that results from the equality (see (2.2) for $v = u_\lambda$)

$$\int_G \lambda A u_\lambda dG = \Phi(u_\lambda)$$

and the estimate (3.6).

Let the approximate solution (3.1) be defined by one approximation $\{u_i, \lambda_j\}$. Then we obtain values of the constants from (3.3) and (3.5):

$$a_i = b_j \frac{\int_G \varphi_j A\varphi_m dG}{\Phi(\varphi_i, \varphi_m)} = b_j c_{jm}, \quad b_j = \frac{\int_G K\varphi_l dG \cdot \Phi(\varphi_i, \varphi_m)}{\int_G A\varphi_i \varphi_l dG \cdot \int_G A\varphi_m \varphi_j dG}$$

It is hence seen that the constants

$$a_i = \int_G K\varphi_l dG \cdot \left(\int_G A\varphi_i \varphi_l dG\right)^{-1}$$

are outwardly identical to the coefficients in the approximate "Ritz" solution of the problem of minimizing the energy functional of elasticity theory boundary value problems /1/. If a system of coordinate functions, orthonormalized "with respect to energy" for the second boundary value problem of elasticity theory /1/ is taken as the system $\{\varphi_i\}_{i=1}^{i=\infty}$ (in this case the functions φ_i are also not subject to any constraints on S), then there will be the relationship

$$\int_G A\varphi_i \varphi_l dG = \begin{cases} 1, & i = l \\ 0, & i \neq l \end{cases}$$

and the algorithm to find the constants a_i, b_j simplifies significantly.

4. We will now use the proposed algorithm to find the saddle point of the Lagrangian $L(u, \lambda)$ approximately. To do this it is necessary to show that $\{u_k, \lambda_n\} \to \{u_0, \lambda_0\}$ as $k, n \to \infty$. Since we have $\operatorname{grad}_u L(u_0, \lambda_0) = 0$ at the saddle point $\{u_0, \lambda_0\}$, then for $v = u - u_0$ (and for $v = u - u_k$) we obtain the two respective equalities from (1.3)

$$\Phi(u_0, u-u_0)=\int_G \lambda_0 A(u-u_0)\,dG, \quad \forall u \in W_2^2(G)$$

$$\Phi(u_k, u-u_k)=\int_G \lambda_n A(u-u_k)\,dG, \quad \forall u \in W_2^2(G)$$

From the first equation for $u = u_k$ and from the second for $u = u_0$, by subtracting one from the other we obtain

$$\Phi(u_0-u_k, u_0-u_k)=\int_G (\lambda_0-\lambda_n) A(u_0-u_k)\,dG\,.$$

Using the estimate (3.6) for the left side of this equality, and the Cauchy inequality for the right side, we obtain

$$c\,\|u_0-u_k\|^2_{W_2^1(G)} \leqslant \|\lambda_0-\lambda_n\|_{L_2(G)}\,\|A(u_0-u_k)\|_{L_2(G)} \leqslant$$
$$\|\lambda_0-\lambda_n\|_{L_2(G)} c_1 \|u_0-u_k\|_{L_2(G)} \leqslant \|\lambda_0-\lambda_n\|_{L_2(G)} \times$$
$$c_1 c_2 \|u_0-u_k\|_{W_2^1(G)}. \quad (c_1=\|A\|_{L_2(G)},\ c_2>0)\,.$$

The estimate from the imbedding theorem $W_2^1(G) \subset L_2(G)$ is used here.

Summarizing, the following inequality holds

$$\|u_0-u_k\|_{W_2^1(G)} \leqslant \frac{c_1c_2}{c}\,\|\lambda_0-\lambda_n\|_{L_2(G)}$$

from which it follows that if the condition $\|\lambda_0-\lambda_n\|_{L_2(G)} \to 0$ is satisfied as $n\to\infty$, then the convergence $\|u_0-u_k\|_{W_2^1(G)} \to 0$ also holds as $k\to\infty$. Therefore, the foundation of the algorithm reduces to proving that the sequence of approximations $\{\lambda_n\}$ minimizes the functional $F(\lambda)$ for which (2.4) is the Euler-Lagrange equation (by virtue of /1/,p.367 the sequence $\{b_j\varphi_j\}_{j=1}^{j=\infty}$ is complete in $L_2(G)$ since completeness of $\{\varphi_j\}$ in $L_2(G)$ is assumed).

For $u_\lambda \equiv u(\lambda)$ the functional

$$F(\lambda)=\int_G \lambda A u_\lambda\,dG - 2\int_G \lambda K\,dG, \quad K \in L_2(G)$$

is a quadratic functional of the vector λ with positive-definite quadratic form

$$\int_G \lambda A u_\lambda\,dG$$

(by virtue of the equality $\int_G \lambda A u_\lambda\,dG = \Phi(u_\lambda)$ and the estimate (3.6)), which for the discretization described above

$$\lambda_n=\sum_{j=1}^{n} b_j\varphi_j$$

is a quadratic form of the coefficients b_j. Then the sequence of approximations $\{\lambda_n\}$ in which the coefficients b_j are the solution of a system of linear equations obtained from the condition

$$dF(\lambda_n)/db_j = 0 \ (j=1,2,\ldots,n)$$

is minimizing for the functional $F(\lambda)$, i.e., $\lim F(\lambda_n)=F(\lambda_0)$ as $n\to\infty$ (/1/,p.98). Therefore, the sequence $\{\lambda_n\}$ converges such that $\|\lambda_0-\lambda_n\|_{L_2(G)} \to 0$ as $n\to\infty$, which also means that $\|u_0-u_k\|_{W_2^1(G)} \to 0$ as $k\to\infty$.

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